De Bruijn-Erdős type theorems for graphs and posets

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Abstract

A classical theorem of De Bruijn and Erdős asserts that any noncollinear set of n points in the plane determines at least n distinct lines. We prove that an analogue of this theorem holds for graphs. Restricting our attention to comparability graphs, we obtain a version of the De Bruijn-Erdős theorem for partially ordered sets (posets). Moreover, in this case, we have an improved bound on the number of lines depending on the height of the poset. The extremal configurations are also determined.

1 Introduction

The starting point of this paper is a classical result of De Bruijn and Erdős in combinatorial geometry. A set of n points in the plane, not all on a line, is called a *near-pencil* if exactly n-1 of the points are collinear.

Theorem 1 Every noncollinear set of n points in the plane determines at least n lines. Moreover, equality occurs if and only if the configuration is a near-pencil.

Erdős [15] showed that this result is a consequence of the Sylvester-Gallai theorem which asserts that every noncollinear set of n points in the plane determines a line containing precisely two points. Later, De Bruijn and Erdős [14] proved a more general combinatorial result which implies Theorem 1. Coxeter [12] showed that the Sylvester-Gallai theorem holds in a more basic setting known as ordered geometry. Here, the notions of distance and angle are not used and, instead, a ternary relation of betweenness is employed. We write [abc] for the statement that b lies between a and c. Using this notation, a line can be defined in a simple way. For any two distinct points a and b, the line \overline{ab} is defined as

$$\overline{ab} = \{a, b\} \cup \{x : [xab] \text{ or } [axb] \text{ or } [abx]\}$$
 (1)

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Therefore, any set of points with a betweenness relation defined on them determines a family of lines. For example, in an arbitrary metric space (V, d), Menger [8] defined betweenness on V in the following natural way:

$$[axb] \Leftrightarrow d(a,x) + d(x,b) = d(a,b).$$

Using (1), the line \overline{ab} can be defined for any two distinct points a and b in V. Observe that this definition of a line generalizes the classical notion of a line in Euclidean space to any metric space. These lines may have strange properties: two lines might have more than one common point, and it is even possible for a line to be a proper subset of another line. A line is called *universal* if it contains all points in V. Chen and Chvátal [6] proposed the following conjecture, which, if true, would give a vast generalization of Theorem 1.

Conjecture 1 (Chen-Chvátal [6]) Any finite metric space on n points either induces at least n distinct lines or contains a universal line.

Although the conjecture has been proved in several special cases [3, 5, 7, 10, 11, 13], it is still wide open. The best known lower bound on the number of lines in a general finite metric space with no universal line is $(1/\sqrt{2} + o(1))\sqrt{n}$ [2].

Recently, Chen and Chvátal [6] generalized the notion of lines in metric spaces to lines in hypergraphs. A hypergraph is an ordered pair (V, \mathcal{E}) such that V is a set of elements called the vertices and \mathcal{E} is a family of subsets of V called the edges. A hypergraph is k-uniform if each of its edges consist of k vertices. They observed that given a metric space (V, d), one can associate a hypergraph $H(d) = (V, \mathcal{E})$ with $\mathcal{E} := \{\{a, b, c\} : [abc] \text{ in } (V, d)\}$. If the line \overline{ab} in the 3-uniform hypergraph is defined as

$$\overline{ab} = \{a, b\} \cup \{x : \{a, b, x\} \in \mathcal{E}\},\$$

then the metric space (V, d) and the hypergraph (V, \mathcal{E}) determine the same set of lines.

They proved that there is an infinite family of 3-uniform hypergraphs inducing only $c^{\sqrt{\log_2 n}}$ distinct lines (where n is the number of vertices and c is a constant). This means that there are infinitely many 3-uniform hypergraphs for which the analogue of Theorem 1 does not hold. However, analogues of Theorem 1 have been shown to hold for some special families of 3-uniform hypergraphs in [4]. The best known lower bound on the number of lines in a 3-uniform hypergraph with no universal line is $(2 - o(1)) \log_2 n$ [1].

Following the lead of these previous works, we obtain an analogue of De Bruijn-Erdős theorem for posets. Let $P = (X, \prec)$ be a finite poset with the order relation \prec defined on the set X. The size of a maximum chain in P is called the *height* of P and is denoted h(P).

As in the metric space case, a poset P induces a natural betweenness relation:

$$[abc] \Leftrightarrow a \prec b \prec c \text{ or } c \prec b \prec a.$$

Therefore, we can again define lines in posets using (1). Observe that, if a is incomparable to b, then the line $\overline{ab} = \{a, b\}$ and, if a is comparable to b, then

$$\overline{ab} = \{a, b\} \cup \{x : x \text{ is comparable to both } a \text{ and } b\}.$$

As before, a line is *universal* if it contains every point from the ground set. Our main result is to show that an analogue of Conjecture 1 holds for posets. In fact, we obtain a stronger bound as a function of the height of the poset.

Theorem 2 If P is a poset on n vertices with no universal line, and $h(P) \ge 2$, then P induces at least

$$h(P)\binom{\lfloor n/h(P)\rfloor}{2} + \lfloor n/h(P)\rfloor (n \bmod h(P)) + h(P)$$
(2)

distinct lines.

Observe that (2) is always greater than or equal to n (with equality if $h(P) \ge \lfloor n/2 \rfloor$). Moreover, if $h(P) = O(n^s)$ for $0 < s \le 1$, the number of distinct lines in P is $\Omega(n^{2-s})$.

Our second result is a generalization from posets to graphs. For any graph G=(V,E) and vertices $a,b\in V$, we can define the line \overline{ab} as

$$\overline{ab} = \{a, b\} \cup \{c : abc \text{ is a triangle}\}.$$

Again, the line \overline{ab} is universal if it contains every vertex in V. We prove

Theorem 3 If a graph G on $n \ge 4$ vertices does not contain a universal line, then it induces at least n distinct lines, and equality occurs only if G consists of a clique of size n-1 and a vertex that has at most one neighbor in the clique.

Remark 1 It may be easily seen that the theorem also holds when n = 3, but we have an additional extremal example in this case: a graph where all pairs of vertices are non-adjacent.

Observe that to any poset $P = (V, \prec)$, we can associate a graph G = (V, E) where $ab \in E$ if and only if $a \prec b$ or $b \prec a$. Such a graph is called a *comparability graph*. Hence, for any three vertices a, b, c of P, we have $a \prec b \prec c$ if and only if abc is a triangle in the corresponding comparability graph. Therefore, the graph case is a generalization of the poset case.

Using Theorem 3, it can be easily seen that, in the case of posets, equality occurs only if the poset consists of a chain of size n-1 and a vertex which is comparable to at most one vertex of this chain.

The paper is organized as follows. In Section 2, we prove Theorem 2 by providing an algorithm for finding lines. In Section 3, we prove Theorem 3 by induction.

2 Lines in Posets

We begin by introducing some notation that will be useful in the proof of Theorem 2. For any pair of elements a, b in a poset $P = (X, \prec)$, we write $a \nsim b$ to indicate that the points a and b are not comparable (that is, neither $a \prec b$ nor $b \prec a$ hold). Let $Y \subseteq X$. We denote by $P \setminus Y$ the poset on the set of points $X \setminus Y$ together with \prec restricted to $X \setminus Y$.

In this section, we prove a lower bound on the number of lines in a poset as a function of its height. Before we proceed with the proof, we need a simple lemma.

Lemma 1 If $A_1, ..., A_r$ are r sets such that $\sum_{i=1}^r |A_i| = n$, then $\sum_{i=1}^r {A_i \choose 2} \ge r {\lfloor n/r \rfloor \choose 2} + \lfloor n/r \rfloor (n \mod r)$.

PROOF — Observe first that if $||A_i| - |A_j|| \le 1$ for all i and j, then the bound holds. Thus, it suffices to prove that if $|A_i| - |A_j| > 1$, then moving one point from A_i to A_j does not increase $\sum_{i=1}^{H} {|A_i| \choose 2}$. Let $x \in A_i \setminus A_j$ and $A'_i = A_i \setminus \{x\}$, $A'_j = A_j \cup \{x\}$. We have

$$\binom{|A_i|}{2} + \binom{|A_j|}{2} \ge \binom{|A_i|-1}{2} + \binom{|A_j|+1}{2} = \binom{|A_i'|}{2} + \binom{|A_j'|}{2}$$

by the convexity of $\binom{m}{k}$ in m, and the lemma follows.

2.1 Proof of Theorem 2

Let \mathcal{A} be a maximal partition of P into antichains, and let $C \subseteq P$ be a maximal chain in P. By Mirsky's Theorem, we know that $|\mathcal{A}| = |C| = h(P)$. For notational convenience, from now on, let h(P) be denoted by H. Denote the elements of \mathcal{A} and C, respectively, as $\mathcal{A} = \{A_1, \ldots, A_H\}$ and $C = \{c_1 \ldots c_H\}$ with $c_1 \prec \cdots \prec c_H$. Assume, without loss of generality, that $c_i \in A_i$ for $i = 1, \ldots, H$.

Set

$$\mathcal{L}_0 := \bigcup_{i=1}^{H} \{ \overline{ab} : a, b \in A_i, a \neq b \}.$$

Note that all of the lines in \mathcal{L}_0 are induced by incomparable points and are, thus, pairwise distinct. By Lemma 1, we have

$$|\mathcal{L}_0| = \sum_{i=1}^H {|A_i| \choose 2} \ge H {\lfloor n/H \rfloor \choose 2} + \lfloor n/H \rfloor (n \bmod H).$$

Next we use the chain C to find H further lines, distinct from those in \mathcal{L}_0 . We do so via the following iterative process:

Set $b_1 = 1$, $t_1 = h$ and $\mathcal{L}_1 = \emptyset$. For k = 1, 2, ..., apply the following steps until a STOP condition is met.

Step 1 If $b_k = t_k$, set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_1 c_H}\}$ and STOP.

Otherwise $b_k < t_k$ and there exists $s_k \notin \overline{c_{b_k} c_{t_k}}$. If s_k is incomparable with both c_{b_k} and c_{t_k} , go to **Step 2a**. If s_k is incomparable with c_{b_k} and comparable with c_{t_k} , go to **Step 2b**. Finally, if s_k is incomparable with c_{t_k} and comparable with c_{b_k} , go to **Step 2c**.

Step 2a Set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \le i \le t_k\} \cup \{\overline{c_1 c_H}\}$ and STOP.

Step 2b Set $b_{k+1} = \max_{b_k \le i < t_k} \{i : c_i \nsim s_k\} + 1$,

$$t_{k+1} = t_k, \mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \le i \le b_{k+1}\}.$$
 Go to **Step 1**.

Observe that, in this case, we have that $b_k < b_{k+1} \le t_k$, $s_k < c_{b_{k+1}}$ and, for $j = b_k, \ldots, b_{k+1} - 1$, we have $s_k \nsim c_j$.

Step 2c Set $t_{k+1} = min_{b_k < i \le t_k} \{i : c_i \nsim s_k\} - 1$,

$$b_{k+1} = b_k$$
, $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : t_{k+1} \le i \le t_k\}$. Go to **Step 1**.

Observe that, in this case, we have that $b_k \leq t_{k+1} < t_k$, $c_{t_{k+1}} \prec s_k$ and, for $j = t_{k+1} + 1, \dots, t_k$, $s_k \nsim c_j$.

Assume that the process stops after K iterations.

For any k < K, in the k^{th} iteration, exactly one line added to \mathcal{L}_k is induced by two comparable points. We call this line l_k . Thus, there are K-1 such lines, namely l_1, \ldots, l_{K-1} . Notice that l_k is either $\overline{c_{b_{k+1}}s_k}$ or $\overline{c_{t_{k+1}}s_k}$. If $l_k = \overline{c_{b_{k+1}}s_k}$, since $s_k \prec c_{b_{k+1}}$, we have $\{c_{b_{k+1}}, c_{b_{k+2}}, \ldots c_{b_K}, c_{t_K}, c_{t_{K-1}}, \ldots, c_{t_1}\} \subseteq \overline{c_{b_{k+1}}s_k}$, and since $c_{b_k} \nsim s_k$, we have $c_{b_k} \notin \overline{c_{b_{k+1}}s_k}$. Similarly, if $l_k = \overline{c_{t_{k+1}}s_k}$ we have $\{c_{b_1}, \ldots, c_{b_K}, c_{t_K}, \ldots, c_{t_{k+1}}\} \subseteq \overline{c_{t_{k+1}}s_k}$ and $c_{t_k} \notin \overline{c_{t_{k+1}}s_k}$. Observe now that the line $\overline{c_1c_H}$, that is added at the K^{th} iteration, contains all points in C. This implies that the lines $l_1, \ldots, l_{K-1}, \overline{c_1c_H}$ are pairwise distinct. Thus, the process finds K pairwise distinct lines which are induced by comparable points. Moreover, since all the lines in \mathcal{L}_0 are induced by incomparable points, none of these K lines belong to \mathcal{L}_0 .

The rest of the lines found by the process are induced by incomparable points. Hence, it remains to prove that H - K of them are pairwise distinct and don't belong to \mathcal{L}_0 .

Let k < K. We claim that \mathcal{L}_k contains at least $b_{k+1} - b_k + t_k - t_{k+1} - 1$ (new) lines that are not in \mathcal{L}_{k-1} . Assume first that, in the k^{th} iteration, lines are added at Step 2b (so $t_k - t_{k+1} = 0$). So $b_{k+1} - b_k$ lines induced by two incomparable points are added, namely $\overline{c_{b_k} s_k}, \ldots, \overline{c_{b_{k+1}-1} s_k}$. At most one of these lines belongs to \mathcal{L}_0 and none of them belongs to $\mathcal{L}_{k-1} \setminus \mathcal{L}_0$ because lines induced by incomparable points that are added in previous iterations of the process, involve points of C either strictly below b_k or strictly above $t_k \geq b_{k+1}$. Hence, at least $b_{k+1} - b_k - 1$ new lines induced by incomparable points are added at Step 2b. A symmetric argument proves that, in the case where the lines are added at Step 2c (so $b_{k+1} - b_k = 0$), we have added $t_k - t_{k+1} - 1$ new lines induced by incomparable points.

So, after K-1 iterations, the number of lines induced by incomparable points, in $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ is

$$\sum_{k=1}^{K-1} (b_{k+1} - b_k + t_k - t_{k+1} - 1) = t_1 - b_1 - (t_K - b_K) - (K - 1) = H - K - (t_K - b_K).$$

Hence, it remains to show that $t_K - b_K$ new distinct lines induced by incomparable points are added at the K^{th} iteration. In the case where $b_K = t_K$ we are done, so we may assume that $b_K < t_K$ and the process terminates at Step 2a. So, the lines $\overline{c_{b_K} s_k}$, $\overline{c_{b_K+1} s_k}$, ..., $\overline{c_{t_K} s_k}$ are added. At most one of these lines belong to \mathcal{L}_0 and none of them belong to $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ since lines induced by incomparable points added at iterations $1, \ldots, K-1$ involve points of C either strictly below b_K or strictly above t_K . It follows that $t_K - b_K$ new lines are added.

3 Lines in Graphs

We first need two easy observations about lines in a graph G = (V, E). A vertex x in a graph is universal if it is adjacent to all vertices in $V \setminus x$.

- 1. If $ab \notin E$, then $\overline{ab} = \{a, b\}$,
- 2. A line \overline{ab} is universal if and only if both a and b are universal.

We are now ready to prove our generalization to the graph case.

3.1 Proof of Theorem 3

We will use induction on n on the full statement of the theorem. First, we show that the theorem holds when n=4. If there is no triangle in our graph, then every pair of vertices induces a distinct line, giving us 6 lines. If there are two different triangles in our graph, then there exist 2 vertices p,q, that belong to both triangles, and the line \overline{pq} is universal, a contradiction. Therefore, we have exactly one triangle, and it is easy to see that in this case we have exactly 4 lines in our graph and the extremal graphs are exactly as desired.

Let G = (V, E) be a graph on $n \ge 5$ vertices having no universal lines, and assume the statement holds for smaller n.

Let $V_1 \subseteq V$ be the set of those points x such that $G \setminus \{x\}$ has a universal line, and set $V_2 = V \setminus V_1$. Assume first that $V_2 = \emptyset$. So $V_1 = V$ and, thus, for any $x \in V$, $V \setminus \{x\}$, induces a universal line. Since G has no universal lines, $V \setminus \{x\}$ is a line of G for any $x \in V$. Thus, G induces n distinct lines of size n-1. Moreover, since it has no universal lines, G has at least two non-adjacent vertices, providing a line of size two. Thus, if $V_2 = \emptyset$, we have that G induces at least n+1 lines.

So we may assume from now on that $V_2 \neq \emptyset$. We will distinguish between two cases:

Case 1 There exists a point x in V_2 that is not universal.

Let y be a vertex non-adjacent to x. Since $G \setminus \{x\}$ has no universal lines, by induction, $G \setminus \{x\}$ induces at least n-1 distinct lines. If ℓ is a line of $G \setminus \{x\}$, then either ℓ or $\ell \cup \{x\}$ is a line of G. It follows that these lines are all distinct in G. Moreover, if they contain x, then they have at least 3 vertices and so they are all distinct from $\overline{xy} = \{x, y\}$. Hence, G has at least n distinct lines.

Now, assume that G induces exactly n distinct lines. Then, $G \setminus \{x\}$ must contain exactly n-1 lines and so by induction, $G \setminus \{x\}$ consists of a clique K on n-2 vertices, $x_1, x_2, \ldots, x_{n-2}$, and a vertex z which has at most one neighbor in K. Notice that the set of lines of $G \setminus \{x\}$ is $\mathcal{L}_{G \setminus \{x\}} := \{\{x_1, \ldots, x_{n-2}\}, \{z, x_1\}, \ldots, \{z, x_{n-2}\}\}$, giving us n-1 distinct lines of G, namely, $\mathcal{L} := \{\ell \text{ or } \ell \cup \{x\} \mid \ell \in \mathcal{L}_{G \setminus \{x\}}\}$.

We claim that x is adjacent to all vertices of $V \setminus \{x,y\}$ because otherwise there exists a vertex y' in $V \setminus \{x,y\}$ such that $\overline{xy'} = \{x,y'\}$ is a line of G. Since $\overline{xy} = \{x,y\}$ is also a line of G, and $\overline{xy}, \overline{xy'} \notin \mathcal{L}$, G induces at least n+1 distinct lines contradicting our assumption.

Assume that $y \in K$, and let z' be the unique neighbor of z in K. Consider a vertex w in $K \setminus \{y, z'\}$ (such a vertex exists because $n \geq 5$). Since $y \notin \overline{xw}$ and $z \notin \overline{xw}$, we have $\overline{xw} \notin \mathcal{L}$. Of course, $\overline{xy} \notin \mathcal{L}$ is a line of G like before. Thus, G induces at least n+1 distinct lines again. Therefore, y = z, and $K \cup \{x\}$ is a clique of G as desired.

Case 2 All points of V_2 are universal.

Since G has no universal line, if follows that V_2 contains exactly one vertex, say x. So $V_1 = V \setminus \{x\}$ and, thus, for any $u \in V_1$, $V \setminus \{u\}$, is a line of G. This yields n-1 lines of size n-1. Moreover, it is easy to see that, since G has no universal lines, it must contain at least two pairs of non-adjacent vertices, providing us with two more distinct lines of size two. Hence, G has at least n+1 distinct lines.

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